Key Developmental Understandings in Mathematics: A Direction for Investigating and Establishing Learning Goals

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Although mathematics educators seem to agree on the importance of teaching mathematics for understanding, what they mean by understanding varies greatly. In this article, I elaborate and exemplify the construct of key developmental understanding to emphasize a particular aspect of teaching for understanding and to offer a construct that could be used to frame the identification of conceptual learning goals in mathematics. The key developmental understanding construct is based on extant empirical and theoretical work. The construct can be used in the context of research and curriculum development. Using a classroom example involving fractions, I illustrate how focusing on key developmental understandings leads to particular, potentially useful types of pedagogical thinking and directions for inquiry.

Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. (National Council of Teachers of Mathematics, 2000, p. 11, emphasis added)

Recent discourse in mathematics education has coalesced around the importance of focusing on and fostering students’ mathematical understanding. This agreement among mathematics educators has led to a commitment to generate new learning goals for students that are less skewed in favor of skill and facts learning and more focused on student thinking. Thompson and Saldanha (2004) asserted that “conditions for improved instruction entail an enduring discussion of what the community

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intends students learn” (p. 110). As part of this discussion, I introduce the construct of key developmental understanding (KDU) to emphasize particular aspects of teaching for understanding and to offer a construct that could be used to frame the identification of conceptual learning goals in mathematics.

The majority of mathematics educators would likely report that they endeavor to promote mathematical understanding. However, what they mean by understanding varies greatly (Sierpinska, 1994). One of the most common uses of understanding is knowing why something is true or appropriate (e.g., why the denominator remains the same when I add two fractions that have like denominators). This is consistent with what Skemp (1976) called relational understanding, knowing both what to do and why. A second common perspective on understanding is characterized by Hiebert and Lefevre’s definition (1986) of conceptual knowledge: “knowledge that is rich in relationships … a network in which the linking relationships are as prominent as the discrete pieces of information” (pp. 3–4). Others have also stressed the connected nature of understanding (cf. Ma, 1999; Secada, 1997). Both of these characterizations of understanding are useful in that they suggest kinds of knowledge that are valued; however, they do not help in identifying critical transitions that are essential for students’ mathematical development. The elaboration of the construct of KDU is meant to address this need and suggest a basis for specifying important learning goals.

In the next section, I use an example involving fractions to distinguish and elaborate the construct of KDU. Please note that in discussing students in terms of their (key developmental) understandings, I am not claiming that these understandings exist in the student; rather, specifying understandings is a way that observers (researchers, teachers) can impose a coherent and potentially useful organization on their experience of students’ actions (including verbalizations) and make distinctions among students’ abilities to engage with particular mathematics. Thus, a claim that a learner has understanding $X$ is a communication about the observer’s current model of the learner.¹ Von Glasersfeld (1995) emphasized,

If these concepts were abstractions from reflection rather than from sensorimotor experiences, it was clear from the outset that whatever inferences could be made about them would contain an element of conjecture. In time, however, the resulting hypothetical model achieves a high degree of plausibility and predictive usefulness. (p. 17)

¹Talking about learners as having understanding $X$ is a convenient way of indicating a distinction in what they seem to bring to the task. It is not meant to indicate that the only possibility is that a learner has or does not have an understanding. An observer could infer that a particular understanding is more mature or complete in one student than in another (both of whom have the understanding to some extent).
AN UNDERSTANDING OF FRACTION

I have often engaged teachers in thinking about what it means to have an understanding of fractions. Typically they respond that it means knowing that a fraction, $\frac{m}{n}$ (my language), means that a whole is divided into $n$ equal parts and that I have $m$ of those parts. Consider this articulation of understanding fractions in light of the following classroom episode.

In a fourth-grade class, I asked the students (a) to use a blue rubber band on their geoboards to make a square of a designated size and then (b) to put a red rubber band around one half of the square. Most of the students divided the square into two congruent rectangles. However, Mary cut the square on the diagonal, making two congruent right triangles. The students were unanimous in asserting that both fit with my request that they show half of the square. Further, they were able to justify that assertion by explaining that each of the parts was one of two equal parts and that the two parts made up the whole.

I then asked, “Is Joe’s [rectangular] half larger? Is Mary’s half larger, or are they the same size?” Approximately a third of the class chose one option; another third, the second option; the last third, the final option. In the subsequent discussion, students defended their answers; however, few students changed their answers as a result of the arguments presented.

I offer an interpretation of these observations and then use this fraction example to ground the discussion of KDUs. The students who argued that the rectangular or triangular half was larger conceive of halves as an arrangement in which a whole is partitioned into two congruent parts. They do not understand that partitioning a whole into two equal parts creates a new unit whose size, relative to the original unit (whole), is determined; that is, they do not understand that one-half indicates a quantity (amount), not just an arrangement. Thus, it is possible for them to correctly identify two differently shaped halves as being one half and yet not recognize that those two halves must be equal to each other. Thompson and Saldanha (2004) identified this limited view of fractions as additive, rather than multiplicative.

Educators who understand a fraction as a quantity find it difficult to conceive of this limited understanding of one half (as an arrangement). The following way of thinking about this limited understanding may help. One can partition a square into two rectangles, with any cut parallel to one of the sides. Any such partition will create two parts that can be compared to each other and that sum to the whole. However, in the case where the partition results in equal parts, an important part–whole relationship is determined (from the perspective of those who understand it)—a new, specified unit of quantity is constituted. That is, the whole is twice the size of either of the equal parts. This special relationship between the part and the whole, created by equal partitioning, is neither obvious nor automatic to the young student who is just beginning to explore fractions. Understanding that equal partitioning creates specific units of quantity is an example of a KDU.
CHARACTERISTICS OF KDUS

I use the aforementioned example of fraction understandings to point to two characteristics of KDUs. A first characteristic is that KDUs involve a conceptual advance on the part of students. By conceptual advance, I mean a change in students’ ability to think about and/or perceive particular mathematical relationships. In the fraction example, understanding that equal partitioning creates specific units of quantity represents a significant advance, allowing students to conceive of and act with fractions in powerful ways. Those who had this KDU considered the question of which half is bigger to be trivial. For those who did not have this KDU, not only did they not know that the two differently shaped halves were necessarily equal, but they also could not follow the reasoning of those who did know.

To further exemplify this principal characteristic of a KDU—that it is a conceptual advance, one that changes students’ ability to think about and/or perceive particular mathematical relationships—consider students who understand a fraction as an arrangement and who are confronted with situations involving improper fractions. These students have an inability to conceptualize improper fractions (Tzur, 1999). They understand fifths as five congruent parts made from a whole and do not know how to think about adding one more fifth (six in all). They tend to refer to the six parts as “6 sixths.” Crucial to conceptualizing 6/5 is an understanding of 1/5 as a quantity. The development of an understanding of a fraction as a quantity, a KDU, is an important goal for mathematics instruction and further research (e.g., teaching experiments).

A second characteristic of a KDU is that students without the knowledge do not tend to acquire it as the result of an explanation or demonstration. That is, the transition requires a building up of the understanding through students’ activity and reflection and usually comes about over multiple experiences. This is not an empirical claim about KDUs; rather, it is an argument that a focus is needed on those understandings whose development tends to require more than an explanation or demonstration.

In the fraction example, I indicate that educators frequently think about one half as one of two equal parts of a whole. Whereas this thinking is a reasonable way to describe one’s idea of one half, the example I present illustrates that such a definition does not capture the key understandings that must be a part of development of a mature concept of fraction. ² Young students can easily learn that the parts must be equal (usually assimilated as “identical” and based on their notion of fair sharing). Because they have whole-number concepts, young students can learn the correspondence between the whole numbers in the numerator and denominator and the number of parts involved. Thus, it is relatively unproblematic for students

²In this article, I use understanding, as in key developmental understanding, to refer to a component of a larger conceptual entity, which I refer to as a concept.
to identify a typical shaded rectangle with appropriate vocabulary (“one-half”) and symbols (“1/2”). None of this represents the transition necessary to think about fractions as a quantity. Fostering development of this KDU is a challenging problem for mathematics educators.

Note that my attempt in introducing the KDU construct is to further elaborate a way of thinking about understanding—thinking in terms of understandings that are critical to the development of important mathematical ideas. The construct of KDU is not meant for sorting understandings into those that are KDUs and those that are not. The degree to which particular understandings are critical to the development of a specific concept can be thought of as falling along a continuum. Further, the importance of particular understandings may be relative to the learning trajectory of the learners; that is, particular understandings may be more important to a particular learning path and less so to another. The KDU construct is intended to focus inquiry on and orient pedagogy to key understandings.

**HOW DOES ONE IDENTIFY A KDU?**

Mathematics educators often cannot identify key mathematical understandings by examining their own mathematical understandings. What were key developmental issues early in the development of their understandings are not apparent as they now look at and through their sophisticated understandings. Indeed, many key understandings develop without students’ awareness that a conceptual advance has taken place. Therefore, I can expect little overlap between the taken-as-shared knowledge that the mathematical community has about particular mathematical ideas and KDUs. For example, the formal definition of a rational number is quite different from an articulation of an understanding of a fraction as a quantity.

One way to identify KDUs is to observe students engaged in mathematical tasks to specify understandings that can account for differences in the actions of different students in response to the same task. A way to explain these observed differences is by postulating a KDU. Clement (2000) emphasized that such explanations, “are not merely condensed summaries of empirical observations but, rather, are inventions that contribute new … concepts that are part of the scientist’s view of the world and that are not ‘given’ in the data” (p. 549).

In the fraction example, contrasting the students who claimed that the triangular half was larger with the students who claimed that the triangular half was equal to the rectangular half creates a context for identifying a key aspect of understanding fractions. Contrasting students engaged in the same task was in fact the strategy of Piaget (1952) and his colleagues, who identified KDUs such as class inclusion and conservation of number and volume.

To summarize, a KDU in mathematics is a conceptual advance that is important to the development of a concept. It identifies a qualitative shift in students’
ability to think about and perceive particular mathematical relationships—in other words, a significant change in the assimilatory structures that students have available. The emphasis here is on ability to think about and perceive. I am not referring to a missing piece of information that affects students’ performance; rather, I am emphasizing that without completing a developmental process, the students lack a particular mathematical ability. One additional point is important to emphasize. KDUs can be specified at different levels of detail. If I return to the example of fraction as an arrangement and fraction as a quantity, I can compare that specification to the more detailed work of Steffe (2002) and Tzur (1999). The level of detail specified for a key understanding is adequate if it serves to guide the effort for which it is needed (e.g., curriculum design, further research).

**KDUS AND A SOCIAL PERSPECTIVE ON LEARNING**

Currently, mathematics education researchers fall into at least three groups: those who adhere to a social perspective, those who adhere to a cognitive perspective, and those who coordinate the use of social and cognitive perspectives. My work falls into this third category. Cobb and his colleagues in the United States and Bauersfeld and his colleagues in Germany (Cobb & Bauersfeld, 1995) have articulated the usefulness of coordinating these two perspectives.

The KDU construct and reflective abstraction, discussed in the next section, derive from a cognitive perspective. However, they do not conflict with social constructs such as negotiation of meaning and social and sociomathematical norms (Yackel & Cobb, 1996). Bereiter (1985) pointed out that social theory fails to explain internalization: “The whole paradox hides in the word ‘internalizes.’ How does internalization take place? It is evident from Luria’s first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved” (p. 206). The idea of coordinating perspectives is an attempt to make use of the tools available in each and, ultimately, to make progress on problems such as the one highlighted by Bereiter.

**THE NATURE OF MATHEMATICAL UNDERSTANDINGS**

KDUs are mathematical understandings. The KDU construct affords a focus on those mathematical understandings that are particularly pivotal in the mathematical development of students. However, an explication of the KDU construct should include an articulation of mathematical understanding upon which the KDU construct is based. Building on ideas of Piaget and others, I consider a mathematical understanding to be a learned anticipation of the logical necessity of a particular pattern or relationship. In examining this claim, I make a distinction between
empirical learning processes and reflective abstraction, a distinction that is a modification of one made by Piaget.

An empirical learning process is an inductive process through which students discover patterns. By inductive process, I mean multiple trials in which students make an input (or observe an input) and then observe an output. Students learn that the pattern exists. The phenomenon that generates the pattern may remain a black box to the students. I have used the term empirical learning process rather than Piaget’s term empirical abstraction to indicate a category that is broader than the one created by Piaget. Piaget (1978, 2001; von Glasersfeld, 1995) used empirical abstraction (also called physical or simple abstraction) to refer to an inductive process that leads to abstraction of properties of physical objects or the material aspects of a physical action. In contrast, I use empirical learning processes to also include processes not based on physical objects or actions. For example, if students multiply different whole numbers by 6 and observe that the products are even, this would be an empirical learning process resulting in the abstraction that multiplying by 6 generates an even product.

Reflective abstraction, according to Piaget (2001), is the process by which higher-level mental structures are developed from lower-level structures, a coordination of actions leading to a new conception. He described reflective abstraction as having two phases: a projection phase, in which the actions at one level become the objects of reflection at the next, and a reflection phase, in which a reorganization takes place. Knowledge of the logical necessity of a particular pattern or relationship is generated through reflective abstraction. By anticipation of the logical necessity, I mean that the activity no longer needs to be carried out—the effect of the activity is anticipated (for an elaboration of reflective abstraction for the purposes of mathematics education, see Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004). As a result of reflective abstraction, the students learn not just that the relationship exists but why the relationship is necessary. To use the example of multiplying by 6, imagine students who are exploring with cubes. They make an even number by creating a set of pairs. They create odd numbers by creating a set of pairs plus an unpaired cube. Through their exploration they can come to anticipate that six even numbers would have all cubes paired (because only paired cubes are used) and that six of an odd number would have six single cubes that could be reorganized into three pairs. Thus, not only would these students know that multiplying by 6 produces an even number, but they would also understand the logical necessity of this phenomenon.

Piaget (1980) emphasized that reflective abstraction “alone supports and animates the immense edifice of logico-mathematical construction” (p. 92). Consistent with this point is that KDUs (and mathematical understandings more generally)

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3I made this distinction in Simon (2003), although at that time I referred to logico-mathematical activity. I now use the term reflective abstraction.
are never the result of empirical learning processes. Consider an empirical learning process in the context of the fraction example.

For the classroom example discussed earlier, some mathematics educators conclude that “the students do not know that one half of a particular whole, no matter what its shape, is always the same size.” This conclusion is essentially a summary of what was observed and does not get at underlying understandings (KDU). Through this conclusion, they advocate a teaching intervention in which the students are encouraged to cut up the triangular half and superimpose it on the rectangular half. This exploration can be expanded to other shapes and other fractions. The expectation would be that the students would eventually conclude that the shape of the fraction is not important, that one half of any shape is the same size. This intervention has many of the features associated with the current mathematics education reform: hands-on problem solving, student exploration, attention to patterns, and students drawing their own conclusions. However, if I focus on the KDU of a fraction as a quantity, I have reason to conclude that the intervention described is inadequate. Consider this conclusion.

The students who originally claimed that the halves were the same did not make that claim because they had cut up a rectangular half previously nor because they were visualizing such a cut. They knew instantly and with certainty that the halves must be the same size. The pedagogical intervention that involves cutting up the triangle is aimed at getting students to know that the two halves are in fact the same size. The intervention does not engender the anticipation that they must be the same size. That is, the intervention does not develop the understanding that equal partitioning results in a new unit that has a fixed relationship to the original unit (whole). Tzur (1999) described a teaching experiment with students aimed at promoting this KDU. Because they focused on a KDU, the interventions are quite different from the fraction-cutting activity described in the preceding paragraph. Briefly, the approach used by Tzur and colleagues involved tasks in which students endeavored to partition wholes into particular numbers of parts. The students did this by estimating the size of the part and iterating the estimated part to see if the part was the appropriate size. If their estimates were too large or too small, they adjusted their estimates and iterated the new estimate. Through such activities with different-size wholes and different numbers of parts, the students were able to come to anticipate that the size of the part is determined by the size of the whole and the number of parts. Note that they did not just know that this is true; rather, they anticipated how a larger piece would iterate to a whole that was greater than the whole that they were working with and vice versa for a smaller piece.

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4Empirical processes do have their place in mathematics education (e.g., the generation of conjectures to be proved). However, empirical processes are not the mechanism by which new understandings are developed.
KEY DEVELOPMENTAL UNDERSTANDINGS

Although the idea of KDUs was not previously made explicit, important examples of KDUs (not identified as such) can be found in the literature. Early number knowledge is one area in which KDUs have been clearly identified. Researchers have identified cardinality, composite units, and conservation of number as KDUs that significantly affect children’s abilities to conceive of and work with numbers (Gelman & Gallistel, 1978; Piaget, 1952; Steffe & Cobb, 1988). Consider composite units as an example. A conceptualization of composite units involves an understanding that single units can be combined to form a new unit that is itself a countable unit. Students with a conceptualization of composite units can see 10 as 10 single units and as 1 composite unit of 10. This understanding is essential for the development of concepts of place value and multiplication—thus, the identification of composite units as a KDU.

Heinz (2000), building on the work of other researchers (e.g., Confrey, 1994; Kaput, 1985; Kieren, 1988; Schwartz, 1988; Thompson, 1994), identified a KDU in the area of proportional reasoning. She described an understanding of ratio as quantity as “the understanding that a ratio is an intensive quantity—a single quantity that measures the multiplicative relationship between two quantities.” This understanding seems to be required for a sophisticated conception of ratio, one that can be used in mathematizing various problem situations, allowing students to identify when a multiplicative comparison is appropriate (as opposed to an additive comparison). A sophisticated understanding of ratio affords an understanding of a number of other concepts, including slope, arithmetic mean, and probability. One could also consider an understanding of distribution to be a KDU. An understanding of distribution involves being able to view a distribution as not only a set of data points but also as a mathematical object that has its own characteristics that can be observed and compared to the characteristics of other distributions (McClain & Cobb, 2001).

Researchers are exploring other conceptual areas that are problematic for students. In some of these areas, it is not clear that KDUs have been identified (e.g., function and statistical variation).

Articulating the KDU construct is meant to stimulate and focus research and to offer a particular link between such research and the development of mathematics curricula. One important step in developing a scientific approach to mathematics teaching and curriculum design is the identification of conceptual learning goals. Thinking in terms of KDUs is one way of conceptualizing such goals. The identification of KDUs by researchers enables a second level of research—namely, exploring how concepts are developed. Identifying KDUs is a way of specifying developmental steps. As KDUs are specified, inquiry into the development from one step to the next becomes possible.
A focus on KDUs represents a significant change from a more conventional focus on students’ knowledge. A focus on KDUs engenders a different set of questions for inquiry and a different direction for instruction and assessment. In the fraction example, I highlighted the contrast between teaching that was based on the KDU of fraction as a quantity and teaching that was not focused on the KDU. In that example, Tzur’s tasks (1999) for fostering the development of the KDU were quite different from the tasks involving cutting up shapes and superimposing them onto others. I now use the fraction example to illustrate the influence of thinking about KDUs in the context of assessment.

If mathematics educators identify understanding of one fourth as knowing that it is one of four equal parts that make up a whole, what kind of assessment might they generate? One possibility is that they create a series of diagrams for which the students must decide if each represents one fourth. Some of the diagrams would have one of four congruent parts shaded. Some would have more than one of the four congruent parts shaded. Some would have a different number of congruent parts, and some would have unequal parts. This assessment is appropriate for the specification of one fourth as one of four equal parts of a whole. That is, the assessment provides evidence of whether the students know that four parts are involved, that they must be equal, and that the attention is focused on one of those parts. However, if one focuses on fraction as a quantity as a KDU, then the assessment item featured in Figure 1 becomes an important one, potentially discriminating between students who understand a fraction as an arrangement and those who understand a fraction as a quantity (I make this claim while acknowledging the weakness of drawing conclusions from a single assessment item). Students who understand fractions as an arrangement will not identify Figure 1 as an example illustrating one fourth, because the parts displayed are not congruent.

![FIGURE 1](image)

Figure 1: Assessment item for fraction as a quantity.
I offer a second example based on the KDU of ratio as a quantity. Heinz (2000) pointed out that different ratio problems can be solved using different conceptions. For example, Problem 1 can be solved using an identical-groups conception.

Problem 1: To make cement, you mix a 5-kg bag of dry cement with 20 L of water. How many liters of water are needed to mix with 40 kg of dry cement?

In an identical-groups conception, students understand that a quality of interest is determined through the association of particular amounts of two quantities (e.g., 5 kg of cement and 20 L of water) and that the quality is invariant for either a multiple of the two associated quantities or a fraction of the associated quantities (e.g., 40 kg of cement and 320 L of water, 2.5 kg of cement and 10 L of water). However, to assess whether students have the KDU of ratio as quantity, a problem such as Problem 2 (following) is important, because it is not likely to trigger students’ identical-groups conception; that is, it is not obvious that there is a “recipe” that can be repeated to make a larger quantity with the same quality. Thus, students without an understanding of ratio as a quantity are likely to invoke an additive comparison for the problem.

Problem 2: A new housing subdivision offers lots of three different sizes: 185 ft by 245 ft (56 m by 75 m), 75 ft by 114 ft (23 m by 35 m), and 455 ft by 508 ft (139 m by 155 m). If you were to view these lots from above, which would appear most square? Which would be least square? Explain your answers.

Curriculum focused on the development of KDUs would also be different from many curricula currently in use. Because a KDU is an understanding that is by definition not generally learned as a result of a teacher’s telling and showing, it challenges the mathematics education community to devise methodologies for fostering such conceptual changes. My colleagues and I have been working on a framework that can guide pedagogical approaches to the development of KDUs (Simon & Tzur, 2004; Simon et al., 2004). The framework is an attempt to elaborate the construct of reflective abstraction for the purposes of mathematics pedagogy—in particular, the design of tasks to foster students’ development of particular mathematical understandings. Based on the framework, tasks are designed to elicit the use of particular goal-directed activities already available to the students, activities that can be the basis of reflective abstraction of a new mathematical understanding.\(^5\)

\(^5\)It is beyond the scope of this article to provide a detailed articulation of this framework (see the cited articles for more information).
CONCLUSION

In this article, I elaborate the construct of KDU to focus attention on particular aspects of mathematical understanding. In postulating the KDU construct, I attempt to abstract from empirical research on students’ mathematics and to build on prior theoretical work, particularly Piaget’s construct of reflective abstraction. KDUs can be viewed as significant landmarks in students’ mathematical development. The identification of a KDU is the result of in-depth inquiry into the source of difference between less able and more able students in the context of particular mathematical content.

Earlier, I highlighted prior characterizations of mathematical understanding, understanding as knowing why and understanding as a rich network of connected ideas. The KDU construct is consistent with these characterizations in that the development of KDUs can contribute to rich networks of ideas, including knowing why. A focus on KDUs is meant to extend discussion beyond characterization of understanding to the identification of understandings that can serve as conceptual learning objectives.

The KDU construct invites a particular focus for research on students’ thinking and learning, a focus on the question, What understandings are critical to the development of particular mathematical ideas, understandings that account for differences between those learners who show evidence of more sophisticated conceptions from those who exhibit less sophisticated conceptions? I argue that these empirical questions can have direct benefit for the identification of instructional goals. Attention to KDUs in curriculum design, supported by increased research-based knowledge of KDUs, can place the pedagogical focus on those mathematical understandings that are most critical to the progress of mathematics students.

REFERENCES


Key Developmental Understandings


