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Connections Between Generalizing and Justifying: Students' Reasoning with Linear Relationships

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Research investigating algebra students' abilities to generalize and justify suggests that they experience difficulty in creating and using appropriate generalizations and proofs. Although the field has documented students' errors, less is known about what students do understand to be general and convincing. This study examines the ways in which seven middle school students generalized and justified while exploring linear functions. Students' generalizations and proof schemes were identified and categorized in order to establish connections between types of generalizations and types of justifications. These connections led to the identification of four mechanisms for change that supported students' engagement in increasingly sophisticated forms of algebraic reasoning: (a) iterative action/reflection cycles, (b) mathematical focus, (c) generalizations that promote deductive reasoning, and (d) influence of deductive reasoning on generalizing.

Key words: Algebra; Conceptual knowledge; Constructivism; Middle grades, 5–8; Learning; Patterns/relationships in mathematics; Proof; Qualitative methods

Generalization and justification are considered essential components of algebraic activity (Blanton & Kaput, 2002; Reid, 2002; Steffe & Izsak, 2002). Researchers have argued that students should develop general connections early as a foundation for algebraic understanding (RAND Mathematics Study Panel, 2002; Steffe & Izsak, 2002), which has led to a wealth of studies focused on the promotion of generalizing activities (Blanton & Kaput, 2000, 2002; Carpenter & Franke, 2001; Carpenter & Levi, 2000; Schliemann, Carraher, & Brizuela, 2001) and to the publication of curricula geared toward promoting generalizing activities (Coxford et al., 1998; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998; McConnell et al., 1998). The growing focus on generalization reflects the belief on the part of researchers that "generalization and formalization are intrinsic to mathematical activity and thinking—they are what make it mathematical" (Kaput, 1999, p. 137).

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Similarly, students' ability to justify their generalizations has been linked to what it means to reason algebraically (Curcio, Nimerofsky, Perez, & Yaloz, 1997; Petocz & Petocz, 1997; Reid, 2002). Blanton and Kaput (2002) argued that "justification in any form is a significant part of algebraic reasoning because it induces a habit of mind whereby one naturally questions and conjectures in order to establish a generalization" (p. 25). Although proof has not historically been a strong focus in algebra, it has garnered more emphasis in recent curricular suggestions (Andrew, 1995; Epp, 1998; Fitzgerald, 1996; NCTM, 2000). These changes reflect the notion that developing students' understanding of justification in middle school, even at rudimentary levels, may ease the transition to more advanced views of proof in secondary school (Knuth & Elliott, 1998).

Studies investigating algebra students' abilities to generalize and justify, however, suggest that students experience difficulty both recognizing and creating correct general statements and proofs (Chazan, 1993; English & Warren, 1995; Kieran, 1992; Knuth, Slaughter, Choppin, & Sutherland, 2002; Lee & Wheeler, 1987). Examinations of students' work with pattern activities in algebra show that although students recognize multiple patterns, they may not attend to those that are algebraically useful or generalizable (Blanton & Kaput, 2002; English & Warren, 1995; Lee, 1996; Lee &Wheeler, 1987; Orton & Orton, 1994; Stacey, 1989). These studies highlight students' tendency to focus on recursive rather than functional relationships, which presents obstacles toward generalizing the arbitrary case (Blanton & Kaput, 2002; Pegg & Redden, 1990; Schliemann et al., 2001; Szombathely & Szarvas, 1998). In addition, the perception of a valid number pattern has not been shown to guarantee the ability to generalize that pattern correctly (English & Warren, 1995; Orton & Orton, 1994; Stacey & MacGregor, 1997). Furthermore, even when students are able to generalize a pattern or a rule, few are able to explain why it occurs (Coe & Ruthven, 1994). In fact, the wealth of studies investigating students' conceptions of proof demonstrate that middle and high school students overwhelmingly rely on examples to justify the truth of statements (Coe & Ruthven, 1994; Koedinger, 1998; Knuth et al., 2002; Usiskin, 1987).

Although research has documented students' difficulties with both generalization and justification, few have devoted significant attention to the interplay between them. Sophisticated algebraic reasoning, however, depends on deep involvement in both activities. As Lannin (2005) stated, "Further examination of the connection that students see between their generalizations and justifications is important because these two components are closely linked" (p. 232). Certainly, the ways in which students generalize will influence the tools that they can bring to bear when justifying their general statements. If a student creates a generalization based solely on empirical patterns, it would not be surprising if her proof were limited to the use of specific examples. In contrast, a student who generalizes from attending to the quantitative relationships (Thompson, 1988) in a problem might have a better chance of producing a more general argument in her justification. Helping students develop their own algebraically powerful generalizations will likely aid in their abilities to construct appropriate proofs.

Furthermore, the connection between generalization and justification is bidirectional— engaging in acts of justification may be as likely to influence students' abilities to generalize as the other way around. Learning mathematics in an environment in which providing justifications for one's generalizations is regularly expected can promote the careful development of generalizations that make sense and can therefore be explained. A classroom focus on justification could also encourage students to conjecture in order to establish generalizations (Blanton & Kaput, 2002). Otte (1994) agreed, emphasizing that one role of proof should be to aid in generalization: "A proof is also expected to generalize, to enrich our intuition, to conquer new objects, on which our mind may subsist" (p. 310). A focus on justification may help students not only better establish conviction in their generalizations but also aid in the development of subsequent, more powerful generalizations.

Because both generalizing and justifying can influence the development of the other, it is important to understand the nature of this interrelated development; an examination of these relationships was the aim of this study. Students' generalizations were categorized into a taxonomy (Ellis, in press), and their justifications were categorized according to Harel and Sowder's (1998) taxonomy of proof schemes. Qualitative analyses of students' generalizations, justifications, and the links between them led to the identification of four mechanisms for cognitive change that supported students' engagement in increasingly sophisticated forms of algebraic reasoning. This article discusses the four mechanisms for change and the nature of their interaction.

THE ACTOR-ORIENTED APPROACH TO GENERALIZATION AND JUSTIFICATION

Although much of the current research examining students' ability to generalize defines a generalization as a mathematical rule about relationships or properties (Carpenter & Franke, 2001; English & Warren, 1995; Lee, 1996), researchers have historically approached generalization from a number of different perspectives. Generalization as the creation of a rule is similar to Peirce's (1878) notion of induction, but it has also been described as the identification of commonalities (Dreyfus, 1991; Kaput, 1999), which shares some features of traditional views of lateral transfer (Anderson, Corbett, Koedinger, & Pelletier, 1995; Bassok & Holyoak, 1993). Generalization has also been viewed as the process of extending or expanding one's range of reasoning beyond the case or cases considered (Dubinsky, 1991; Harel & Tall, 1991), which has been connected to the process of abstraction (Piaget, 2001). Kaput (1999) described generalization as "lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves, but rather on the patterns, procedures, structures, and the relations across and among them" (p. 137).

An increasing number of researchers have also explored how generalization is distributed across multiple agents set in a specific social and mathematical context (Davydov, 1990; Jurow, 2004; Lobato, Ellis, and Muñoz, 2003). For instance, Reid

(2002) suggested that generalization should be viewed as a collective construct, whereas others argue that generalization cannot be considered in absence of the social, historical, and mathematical context in which it occurs (Dreyfus, Hershkowitz, & Schwarz, 2001; Radford, 1996; Tuomi-Gröhn & Engeström, 2003). These approaches represent a move away from a view of generalization and transfer as an individual, cognitive phenomenon (Greeno, Smith, & Moore, 1993), which could fail to take into account other material, cultural, and social agents.

In response to the limitations of traditional views of transfer (described by Singley & Anderson, 1989), in which transfer is the application of knowledge learned in one situation to another situation, Lobato (2003) developed the actor-oriented transfer perspective. Actor-oriented transfer is a framework that advocates for a shift from the observer's orientation to the actor's orientation when studying processes of transfer. Under this framework, the researcher abandons normative notions of what should count as transfer and instead seeks to understand the processes by which learners generate their own relations of similarity, regardless of their correctness. This allows the knowledge that is generalized to become an object of investigation; when no longer constrained by predetermined criteria for what should constitute transfer, the researcher is free to focus on what appear to be salient features from the student's point of view.

Traditional transfer models employ a static metaphor, in which knowledge is viewed as unchanged during transfer. This metaphor, however, fails to account for cases in which students actively transform their environments into something similar to a known situation (Bransford & Schwartz, 1999; Lobato & Siebert, 2002). In contrast, the actor-oriented perspective views transfer as a dynamic process of creating relations of similarity. Evidence of transfer is therefore provided by either scrutinizing a given activity for any indication of influence from previous activities or by examining how students construe situations as similar (Lobato, in press). By viewing transfer as a dynamic, student-centered process distributed across multiple agents, the actor-oriented perspective provides a mechanism to explain how features of instructional environments may influence students' generalizating (Lobato et al., 2003).

Generalization Taxonomy

Ellis' (in press) generalization taxonomy extended the actor-oriented perspective to describe the different types of generalizations that students create when reasoning algebraically. In the spirit of Kaput's (1999) view, generalization is defined as engaging in at least one of three activities: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases. The actor-oriented perspective guided the development of this definition in two ways. First, generalization is viewed as a dynamic rather than a static process. Second, evidence for generalization is not predetermined but instead is found by identifying the similarities and extensions that *students* perceive as general. This view deviates somewhat from the typical approach to generalization in which a formal verbal or algebraic description of a correct rule is required as evidence of generalization (Orton & Orton, 1994; Stacey & MacGregor, 1997). Ellis' generalization taxonomy emerged from a view that acknowledges the importance of mathematical correctness but also values the need to understand what students themselves see as general.

The taxonomy accounts for multiple levels of generalizing and distinguishes between students' activity as they generalize, called generalizing actions (see Figure 1), and students' final statements of generalization, called reflection generalizations (see Figure 2). Generalizing actions fall into three major categories. When *relating*, students form an association between two or more problems, situations, ideas, or mathematical objects. They relate by recalling a prior situation, inventing a new one, or focusing on similar properties or forms of mathematical objects. When *searching*, students engage in a repeated mathematical action, such as calculating a ratio or locating a pattern, in order to locate an element of similarity. Students focus on relationships, procedures, patterns, or solutions when searching. Finally, *extending* involves the expansion of a pattern, relationship, or rule into a more general structure. Students who extend widen their reasoning beyond the problem, situation, or case in which it originated.[1]

Reflection generalizations represent one's ability to either identify or use an existing generalization. They are final statements of generalization (verbal or written) or the use of the result of a prior generalization. Students' statements of generalization take the form of *identifications* or *statements* of general patterns, properties, rules, or common elements. They also include *definitions* of classes of objects, in which students make statements conveying the fundamental character of a pattern, relationship, class, or other phenomenon. In addition, cases in which students implement previously developed generalizations in new problems or contexts are also categorized as reflection generalizations under the third subcategory, *influence*. Although the three categories of generalizing actions could be viewed in a hierarchical manner because of the increasingly goal-oriented nature represented by movement from Type I (*relating*) to Type II (*searching*) to Type III (*extending*), the three categories of reflection generalizations are not necessarily hierarchical.[2]

Proof Schemes

Harel and Sowder's (1998; Harel, 2006) framework for identifying students' proof schemes is also compatible with the actor-oriented perspective, in that the framework does not represent a researcher's or mathematician's classification of proof content or proof method. Instead it categorizes students' individual schemes of

[1] Although the definition of generalization stated above guided the type of student actions that constituted evidence of generalizing, the categories of generalizing actions do not mirror the definition. Instead, these categories identify a broad range of student actions while generalizing.

[2] For a full description of the taxonomy, see Ellis (in press).

Type I: Relating		Examples
Relating situations: The formation of an association between two or more problems or situations	*Connecting back:* Connecting between a current and previously encountered situation	Realizing that "This gear problem is just like the swimming laps problem we did in class!"
	Creating new: Inventing a new situation viewed as similar to an existing one	"He's walking 5 cm every 2 s. It'd be like a heart that was beating at a steady pace, 5 beats in 2 s."
Relating objects: The formation of an association of similarity between two or more present objects	*Property:* Associating objects by focusing on a property similar to both	Noticing that two equations in different forms both show a multiplicative relationship between x and y
	Form: Associating objects by focusing on their similar form	Noticing that "Those equations both have one thing divided by another"
Type II: Searching		**Examples**
Same relationship: Performing a repeated action in order to detect a stable relationship between two or more objects		Dividing y by x for each ordered pair in a table to determine if the ratio is stable
Same procedure: Repeatedly performing a procedure in order to test whether it remains valid for all cases		Dividing y by x as above without understanding what relationship is revealed by division; dividing as an arithmetic procedure to determine whether the resulting answer is the same
Same pattern: Checking whether a detected pattern remains stable across all cases		Given a table of ordered pairs, noticing that the y-value increases by 7 for each successive pair
Same solution or result: Performing a repeated action in order to determine if the outcome of the action is identical every time		Given an equation such as $y = 2x$, substituting multiple integers for x and noticing that y is always even
Type III: Extending		**Examples**
Expanding the range of applicability: Applying a phenomenon to a larger range of cases than that from which it originated		Having found that the difference between successive y-values is constant for $y = \text{m}x$ equations, applying the same rule to $y = \text{m}x + \text{b}$ equations
Removing particulars: Removing some contextual details in order to develop a global case.		Given that an increase of 1 for x results in an increase of 6/5 for y, and an increase of 1 for y results in an increase of 5/6 for x, developing a general description of this inverse relationship for all linear functions
Operating: Mathematically operating upon an object in order to generate new cases		Knowing that y increases by 6 for every unit increase for x, halving the (1:6) ratio to create a new ordered pair
Continuing: Repeating an existing pattern in order to generate new cases		Knowing that y increases by 6 for every unit increase for x, continuing the (1:6) ratio to create new ordered pairs

Note. A searching action is coded as a relationship or a procedure based on the researcher's understanding of the student's understanding. One can perform the same calculational action in both cases, but the meaning of that action for the student determines whether she is searching for a relationship or performing a procedure.

Figure 1. The generalizing actions of the generalization taxonomy.

Type IV: Identification or statement		Example
Continuing phenomenon: Identification of a dynamic property extending beyond a specific instance		"Every time x goes up 1, y goes up 5" or "For every second, he walks 2/3 cm."
Sameness: A statement of commonality or similarity	*Common property:* Identification of the property common to objects or situations	"For each pair, x is a third of y."
	Objects or representations: Identification of objects as similar or identical	"Even though those equations look different, they're both relating distance and time."
	Situations: Identification of situations as similar or identical	"This gear problem is just like the swimming laps problem we did in class!"
General principle: A statement of a general phenomenon	*Rule:* Description of a general formula or fact	$s \cdot (2/3) = b$, or "You multiply the number the small gear turns by 2/3 to get the number the big gear turns."
	Pattern: Identification of a general pattern	"On the x side it's going up by 1's, and on the y side it's going up by 7's."
	Strategy or procedure: Description of a method extending beyond a specific case	"To find out if each pair represents the same speed, divide miles by hours and see if the ratio is always the same."
	Global rule: Statement of the meaning of an object or idea	"If the rate of change stays the same, the data are linear."
Type V: Definition		**Example**
Class of objects: Definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon		"Any two gears with a 2:3 ratio of teeth will also have a 2:3 ratio of revolutions."
Type VI: Influence		**Examples**
Prior idea or strategy: Implementation of a previously developed generalization		"You could do the same thing on this speed problem that I did with the gears. Divide y by x, and you should get the same thing."
Modified idea or strategy: Adaptation of an existing generalization to apply to a new problem or situation		"Dividing y by x each time doesn't work on this problem, but you could divide the increase in y by the increase in x instead."

Note. Many reflection generalizations mirror generalizing actions. For example, statements of sameness often accompany the generalizing action of relating, and statements of general principles often accompany the generalizing action of searching. The actions of noticing similarity, generating analogous situations, or searching for similarity frequently result in declarations of sameness or articulations of rules and principles. These declarations are the reflection generalizations, and the acts that led to them are the generalizing actions.

Figure 2. The reflection generalizations of the generalization taxonomy.

doubts, truths, and convictions. Defining proving as a process of removing or creating doubts about the truth of an observation, Harel and Sowder distinguish between *ascertaining*, in which one removes his or her own doubts, and *persuading*, in which one removes others' doubts. The processes of ascertaining and persuading are compatible with the notion of justification as discussed in this article. In a manner consistent with the actor-oriented view, the framework seeks to identify what students viewed as convincing: "A person's proof scheme, therefore, is determined chiefly by what constitutes ascertaining and persuading for that person" (Harel, 2006, p. 3).

Five proof schemes in Harel and Sowder's framework applied to the students in the study reported here (see Figure 3). The first two proof schemes, *authoritarian* and *symbolic*, fall under the external conviction family. Under these schemes, conviction is obtained by the word of an authority, or the symbolic form of an argument. Under the empirical family of proof schemes, conjectures are validated or invalidated by specific cases (*inductive*) or sensory experiences (*perceptual*). The final proof scheme, called *transformational*, falls under the deductive category because it includes the validation of a conjecture by means of logical deductions.

The results reported in this article rely on both the generalization taxonomy and the taxonomy of proof schemes. Students' generalizations and justifications were identified and coded according to the taxonomies so that relationships between types of generalizations and types of justifications could be identified.

METHODS

The data reported in this article were gathered in a teaching experiment (Cobb & Steffe, 1983) that occurred at a public middle school located near a large southwestern city. The school had an ethnically diverse population; out of its 1,000 students, approximately 40.8% were Hispanic, 28.2% were Caucasian, 16.7% were Filipino, 6.9% were African American, 6.3% were Asian American, 0.7% were Pacific Islander, and 0.4% were American Indian. Approximately 15% of the students were English language learners.

Participants

Seventh-grade prealgebra students (age 12) were recruited on the basis of willingness to participate in supplemental mathematics lessons, regular classroom attendance, ability to verbalize their thought processes, and grades of C or higher in their mathematics classes. A sample of students who demonstrated an interest in mathematics and an ability to articulate their thoughts was important to the success of the teaching experiment; given the study's goal of tracking students' developing generalizations and justifications, the participants needed to be able to make new connections and describe their thinking. Every student who volunteered for the study was accepted, which resulted in a sample of 6 females and 1 male. Three students were Hispanic, 3 were Caucasian, and 1 was Asian-American.

External conviction	Examples
Authoritarian: The main source of conviction is a statement made by a teacher or appearing in a text.	*Student:* [Looking at a table of data.] Since the *y*-values increase by the same number each time, this will be a straight line. *Interviewer:* Why? *Student:* That's what Ms. R told us.
Symbolic: Students view and manipulate symbols without reference to any functional or quantitative reference.	[Given three connected gears that rotate 6, 4, and 3 times, respectively, students decide that another triple could be 12, 8, and 6 rotations.] *Teacher*: Why is that valid? *Student*: There's a pattern in all of them. So if you do one thing to the small one, you have to do it to the middle one and the big one to keep the ratios the same. *Teacher:* Why does that work? *Student:* It's kind of like changing fractions from 1/2 to 3/6. It's the same thing, just in different form.
Empirical	Examples
Inductive: Students ascertain and persuade by quantitatively evaluating (directly measuring, calculating, substituting specific numbers into expressions, etc.) a conjecture in one or more specific cases	*Student 1:* All of these pairs must be the same speed because cross-multiplying gives the same answer each time. *Teacher:* How do you know that means they're the same speed? *Student 1:* Because 27 times 5 and 7 and 1/2 times 18 equals the same thing. *Student 2:* I tried it for all the pairs in the table and it works every time.
Perceptual: The source of conviction lies in perceptual observations by means of rudimentary mental images. "The important characteristic of rudimentary mental images is that they *ignore transformations on objects or are incapable of anticipating results of transformations completely and accurately*" (Harel & Sowder, 1998, p. 255, emphasis original).	[The student examined a $y = mx$ table of ordered pairs: (2, 9); (5, 22.5); (12, 54); (16, 72).] *Student:* It couldn't be a straight line. *Interviewer:* Why not? *Student:* If you made your graph, it doesn't look like it'd be a straight line. [Sketching a graph that appears curved], 18's up here, 9's like right here, 2 and 22.5 that's like right there, and then 16, 72 would be all the way up there.

Deductive	Example
Transformational: "The transformational proof scheme is characterized by (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory, and (c) transformations of images as part of a deductive process" (Harel & Sowder, 1998, p. 261). Transformations involve goal-oriented operations on mental objects and the ability to anticipate the results of those operations.	[A student explains why he thinks (3/4)$m = b$ should describe the relationship between two gears with 12 and 16 teeth.] *Student:* Since there's 3/4 of the teeth on the small one, the big one always has 1/4 teeth to make up every turn. Making it, the big one turns 3/4 of a turn every time the small one turns once. And so, say it went through 12 teeth on the small gear and 12 on the big gear. That's only 3/4 of a turn for the big gear, while it's a full turn for the small gear. The student could mentally rotate the gears in coordination, matching teeth to teeth, and then multiplicatively compare the remaining teeth to the total teeth. He operated on the gears and their rotations and could anticipate the results of those operations. Although he provided a specific example, he also understood that the ratio would remain constant for any number of turns.

Figure 3. Relevant proof schemes from Harel and Sowder (1998).

One student was an English language learner. Gender-preserving pseudonyms were used for all participants.

The Teaching Experiment

All class sessions during the teaching experiment were videotaped and transcribed. The sessions, which occurred on 15 consecutive school days for 1.5 hours each day, were taught by the author and observed by an assistant who operated the video camera and took detailed field notes. Each session was followed by a 30-minute individual interview with one student, resulting in two interviews per student for the duration of the teaching experiment. The interviews sought to probe more deeply into what students might have generalized or how they might have understood particular concepts addressed on that particular day. The interviews were also videotaped and transcribed.

The goal of the teaching experiment was to explore the ways in which generalizing and justifying activities are related to one another as students meaningfully engage with new mathematical ideas. Grounded in the belief that meaningful engagement can be supported by reasoning with quantitative referents (Thompson, 1988), the sessions focused on exploring linear relationships in two real-world situations: gear ratios and speed. For the first 7 days, the students worked with physical gears to explore gear ratios. For the remaining 8 days, the students worked with the speed simulation computer program SimCalc Mathworlds (Roschelle & Kaput, 1996), with which students could generate and test conjectures about how different combinations of distance and time affected the characters' walking speed.

One aim of the sessions was to help students build from situations in which they could reason quantitatively to develop linearity, rather than inferring linearity from empirical evidence. The learning goals of the teaching experiment therefore included the development of multiplicative ratios, the creation of an emergent linear quantity as the ratio of two initial quantities, and the identification of linear situations as those that have constant ratios. Figure 4 provides an overview of the topics and ideas addressed during the teaching experiment, and Figure 5 provides a sample of the types of problems that students encountered.

Data Analysis

Analysis of the data followed the interpretive technique in which the categories of types of generalizations were induced from the data (Glaser & Strauss, 1967; Strauss & Corbin, 1990). Transcripts of the lessons and interviews were first coded via open coding, in which instances of generalization were identified as they fit the definition of engaging in at least one of three activities: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases. Through multiple passes through the data set, categories of types of generalizations emerged, were tested and modified, and were ultimately formalized into the taxonomy outlined in

Day	Mathematical topics	Class activities	Context
1	Coordinating quantities	Finding ways to keep track of simultaneous rotations of different-sized gears	Gear ratios
2	Relating teeth to rotations; inverse relationships	Determining how to relate the turns of a gear with 8 teeth to a gear with 12 teeth	
3	Constructing ratios; constant ratios in non-uniform tables	Finding relationships between 8/12/16 gears; determining if rotation pairs come from the same gear pair	
4	Connecting $y = ax$ equations to the gear situation	Explaining how $(3/4)m = b$ relates to both rotations and teeth	
5	$y = ax + b$ gear situations	Modeling situations in which A turns before connecting to B	
6	Representing $y = ax + b$ situations in tables	Making $y = ax + b$ tables; comparing and contrasting to $y = ax$ tables	
7	Nonuniform $y = ax + b$ tables; isolating quantities for speed	Determining constant ratio from $y = ax + b$ tables; who walks faster, Clown or Frog	Gear ratios/ speed
8	Changing initial quantities without changing the emergent quantity	Finding as many ways as possible to make Frog walk the same speed as Clown	Speed
9	Classes of equivalent ratios	Explaining why equivalent ratios mean the same speed	
10	Constant ratios in non-uniform tables	Determining if Frog went the same speed, given values in tables	
11	Connecting $y = ax$ equations to the speed situation	Explaining how $(2/3)c = s$ represents speed	
12	$y = ax + b$ speed situations and tables	Modeling situations in which Clown starts away from home and walks a constant speed; making tables to represent $y = ax + b$	
13	Nonuniform $y = ax + b$ tables	Deciding constant speed from $y = ax + b$ tables	
14	Nonuniform $y = ax + b$ tables	Deciding constant speed from tables; describing $y = ax + b$ speed situations	
15	Meaning of linearity	Inventing situations involving linear relationships	Student invented

Figure 4. Overview of the teaching experiment sessions.

Figures 1 and 2. All of the detected generalizations were ultimately coded with the final taxonomy.

Review of the entire data set revealed major trends in the growth of students' generalizing. Early in the sessions, students focused on generalizing from immediate relationships between quantities, whereas in the later sessions, they general-

Connected Gears Problem

You have 2 gears on your table, one with 8 teeth and one with 12 teeth. Answer the following questions:

1. If you turn the small gear a certain number of times, does the big gear turn more revolutions, fewer, or the same amount? How can you tell?
2. Devise a way to keep track of how many revolutions the small gear makes. Devise a way to keep track of the revolutions the big gear makes. How can you keep track of both at the same time?
3. How many times will the small gear turn if the big gear turns 64 times? How many times will the big gear turn if the small gear turns 192 times?

Frog Walking Problem

The table shows some of the distances and times that Frog traveled. Is he going the same speed the whole time, or is he speeding up or slowing down? How can you tell?

Distance	Time
3.75 cm	1.5 sec
7.5 cm	3 sec
12 cm	4.8 sec
15 cm	6 sec
40 cm	16 sec

Figure 5. Sample teaching-experiment problems.

ized across different quantitative situations in order to establish more global rules about linearity. In addition, students' justifications evolved over time from those that were symbolic and empirical to those that were transformational. In attempting to capture the nature of increased sophistication, students' later generalizations and justifications were contrasted to those that they had produced earlier. The four mechanisms for change emerged as a way to explain this growth in sophistication.

These mechanisms were therefore not a priori hypotheses but were instead developed through an examination of the times when students' generalizations and justifications demonstrated a shift in reasoning. When analyzing students' justifications, this meant instances in which students shifted from operating within nontransformational proof schemes to operating within the transformational proof scheme, which was the only proof scheme with which students justified by deductive reasoning. When analyzing students' generalizations, three criteria were employed to assess shifts in reasoning (see Ellis, in press): (a) movement from Type I (*relating*) to Type II (*searching*) to Type III (*extending*) actions, (b) the generation of new inferences, and (c) support for justifications tied to the transformational proof scheme.

The first criterion reflects the increasingly goal-oriented and creative nature of students' reasoning as they moved from *relating* to *searching* to *extending*. *Relating* represented connections that were often spontaneous or not necessarily deliberate or intentional. In contrast, *searching* was a deliberate action that involved both the anticipation of the existence of a relation of similarity and the coordinated effort to locate it. Unlike *relating* and *searching*, *extending* involved reasoning about objects that were not present, which supported the generation of new knowledge.

The second criterion developed because repeated cycles of generalizing produced new understanding as well as the creation of new ideas. Through generalizing, students may develop fresh insight into a problem situation. Thus, if the evolution of a student's generalizations resulted in attention to a different aspect of a problem, the creation of a previously undeveloped idea or a more inclusive connection, or the promotion of a more complex and nuanced understanding, his or her generalizing activity was considered to have grown in sophistication.

The final criterion represents the belief that generalizations for which students can provide justifications via the transformational proof scheme will reflect more sophisticated knowledge than generalizations for which students cannot provide such arguments. Because transformational justifications often emerged over time, it was possible to examine the types of generalizations that ultimately supported those justifications once they emerged. Given the importance of both generalizing and justifying in promoting students' algebraic development, an emphasis was placed on connecting these activities whenever possible throughout the teaching experiment. The generalizations that the students could justify via the transformational proof scheme turned out to be the ones that were robust and well connected to other knowledge.

Once shifts in students' reasoning were identified, the data were then examined to determine the sources of those shifts. Drawing on the constant comparative method (Glaser & Strauss, 1967), conjectures for those sources were developed and revised against new analysis passes. This process led to the four mechanisms discussed below.

RESULTS

The four mechanisms for change are first defined and then illustrated via three data episodes. The episodes are excerpts from a 2-day session in which students identified equivalent speed ratios, and then struggled to explain why those ratios represented the same speed. They form a coherent narrative that demonstrates the four mechanisms interacting to support increasingly sophisticated generalizations and justifications. After each episode is presented, a discussion identifies the mechanisms that promoted progress in students' reasoning.

The Four Mechanisms for Change

Results from the teaching experiment suggest that relationships between generalizing and justifying were rarely self-contained. Namely, the students did not generalize in one way, provide a particular type of justification for that generalization, and then move on. Instead, students built on their prior generalizing and justifying activity in ways that revealed generalizations of an increasingly sophisticated nature over time. Correspondingly, students' justifications grew more sophisticated as well. The four mechanisms emerged as a way to describe the manner in which generalizing and justifying mutually influence one another to support the development of more sophisticated reasoning.

Mechanism 1: Action/Reflection

Students demonstrated iterative action/reflection cycles: They engaged in particular generalizing actions, formalized them as reflection generalizations, and then moved on to new generalizing actions. Although students' initial generalizing actions and associated reflections were frequently limited or even incorrect, subsequent cycles built on previous attempts to develop more sophisticated generalizations. This action/reflection cycle of generalizing constitutes the first mechanism. The action/reflection mechanism works in concert with the other three mechanisms to account for how generalizing and justifying can mutually influence the development of the other.

Mechanism 2: Focus

The mathematical aspects of the problems on which students focused their attention affected their generalizing and justifying. The term *focus* refers to the students' focus, either individually or collectively, rather than the mathematical focus engineered by the teacher or a particular problem situation. For example, all the problems introduced in the teaching-experiment sessions reflected an attempt to place students in quantitatively rich situations. The hope was that the students would explore which quantities affected particular attributes, such as speed and gear ratios, and ultimately develop an understanding of linearity as the constant ratio of the change in one quantity to the change in another. However, there were times when students focused their attention exclusively on number patterns divorced from any quantitative referents. At other times, students attended closely to the quantitative referents, using them to support their reasoning and explanations.

The data suggest that students' focus affected both how they generalized and how they justified. Two relationships between focus and generalizing/justifying emerged. First, when students focused on number patterns, they demonstrated the generalizing actions of *searching for patterns* and *extending by continuing*, and the reflection generalizations of *statements of principles* related to patterns. When an associated justification was found, these generalizations were justified with the external symbolic and empirical proof schemes 71% of the time (20 out of 28). Second, a focus on quantitative relationships was tied to the generalizing action of *searching for the same relationship*, and the reflection generalizations of *statements of continuing phenomena* and *statements of general principles* related to quantities. Associated justifications for these generalizations used the transformational proof scheme 67% of the time (46 out of 69). These two results suggest that the mathematical properties to which the students attended served to either inhibit or promote a growth in sophistication of their reasoning.

Mechanism 3: Generalizations That Promote Deductive Reasoning

The transformational proof scheme was the one proof scheme in which students validated conjectures by means of logical deductions. Those who operated with the transformational proof scheme could (a) consider the generality aspects of an observation, (b) apply goal-oriented and anticipatory mental operations, and (c) transform mental images as part of their deduction processes (for a complete discussion of the transformational proof scheme, see Harel & Sowder, 1998). The development of the transformational proof scheme was one of the aims of the teaching experiment; it represented a shift away from the belief that it is appropriate to justify conjectures by means of examples or external authority. This mechanism addresses the types of generalizing that promoted the transformational proof scheme.

Three types of generalizations were connected to the use of the transformational proof scheme: (a) the generalizing action of *searching for the same relationship* (9 out of 18 associated justifications were transformational), (b) the generalizing action of *extending* (27 out of 45 associated justifications were transformational), and (c) the reflection generalization of a *statement of a continuing phenomenon* (23 out of 31 associated justifications were transformational). Given that the percentage of transformational justifications as a whole was less than 50 (106 out of 216 coded instances of justification), the generalizations that were tied to higher rates of the transformational proof scheme merited special consideration.

Mechanism 4: Influence of Deductive Reasoning on Generalizing

The final mechanism addresses the role of the transformational proof scheme in promoting more powerful generalizations. This finding suggests that students can begin with generalizations that may be limited or unhelpful, but after justifying with the transformational proof scheme, they may subsequently create more accurate, sophisticated generalizations. Results indicate that the transformational proof scheme appeared to promote shifts toward two major types of reflection generalizations: (a) *statements of continuing phenomena*, demonstrating that this relationship is bi-directional; and (b) new *general principles* such as general rules, patterns, and global rules (23 out of the 30 coded generalizations that followed a justification with the transformational proof scheme fell into these two categories; 15 were statements of continuing phenomena, and 8 were new general principles).

Episodes on Equivalent Speed Ratios

The three episodes span 2 days in the teaching experiment. They are presented in narrative form in order to show the mutual influence of generalization and justification on the evolution of students’ reasoning. The mechanisms of change are discussed after each episode.

Episode 1: The Struggle to Explain Equivalent Ratios

This episode demonstrates the influence of the first two mechanisms, action/reflection and focus. The students began the lesson by working on the following problem: “Say Clown walks 15 cm in 12 seconds. Find as many different ways as you can to make Frog walk the same speed as Clown.” Frog and Clown are characters from SimCalc Mathworlds (Roschelle & Kaput, 1996). The students broke into two groups, and both groups decided to develop and test same-speed pairs by running computer simulations. They tested a number of different pairs on a trial-and-error basis by typing various speeds into the computer and watching Clown and Frog walk across the screen. The students then created tables like the one shown in Figure 6.

Larissa, Maria, Dani, and Julie created a shorthand 15:12 unit on their papers. They first multiplied the 15 cm:12 s unit by 2 to obtain 30 cm in 24 s, and they then began to multiply the 15:12 unit by other numbers to create other same-speed pairs. The students engaged in the generalizing action of *extending* by operating on the 15:12 unit, because they operated on the pair to create new instantiations of the ratio. The students then determined that they could multiply the 15:12 unit by any whole number to obtain a new same-speed pair. This resulted in the reflection generalization of the *definition* of a class of same-speed pairs.

Cm	S	yes/NO
15	12	yes
7.5	6	yes
20	4	no
30	25	no
30	15	no
30	24	yes
10	10	no
10	7	no
10	8	yes
11	9	no
11	8.5	no
20	16	yes
60	48	yes
5	4	yes

Figure 6. Students’ table testing distance/time pairs.

The students then realized that all multiples of 5 cm in 4 s would work, as evidenced by Maria's written comment: "Multiple of 5 & 4 has to be a form of 15 cm & 12 seconds." Their recognition of the 5 cm:4 s pair likely occurred as a result of their prior experiences with gears. Every time the students encountered a nonuniform table of pairs of gear rotations, they focused on the smallest whole-number pair they could find in each table. Eventually, they generalized that this pair represented the gear ratio. The students' fixation on the smallest whole-number pair, in combination with other studies showing that students in speed situations do not behave this way (Lobato & Siebert, 2002; Lobato & Thanheiser, 2000, 2002), suggests the influence of prior reasoning with gears. If so, we see here another reflection generalization, the *influence* of a prior idea on the speed situation. Because the microphone was not recording the girls' conversation at this time, there is not enough evidence to determine if they consciously performed the generalizing action of *relating* by connecting back to the gears situation. However, Larissa's language below suggests that at least for her, she connected back to the gears. This connection to the gears situation appeared to help the girls refine their *definition* of a class of same-speed pairs to a larger, more accurate class.

Dani clarified the group's reasoning when the class reconvened as a whole:

Dani: Um, when we figured out that for the thing, it was multiples of 5 and 4.

Teacher: Multiples of 5 and 4. Larissa?

Larissa: Remember when I said that with the gears you switch 'em around and you do the opposite? For this one it works too. Remember the thing I said where you do the 6 and the 5 and you switch 'em and you get 5/6,[3] and that's the number? It works here too.

Larissa was referring to the fact that in every gear table she had seen or created, one of the pairs always represented the gear ratio in the form of two relatively prime whole numbers. She mentioned switching the values because if the gear ratio was, for example, 5/6 (the form students preferred to use, although they understood that 5:6 was equivalent to 1:5/6), then the small gear would rotate six times, while the big gear turned five times. In such cases, the table entry would read Small: 6 :: Big: 5, which Larissa viewed as the "opposite" of 5/6 and in need of reversing. Thus, Larissa demonstrated the generalizing action of *relating* by connecting back to the gear situation, and she produced the reflection generalization of the *identification* of the common property in both types of tables. Although Larissa's connection is not mathematically relevant to the phenomena of gear rotations or speed, under the actor-oriented perspective it constitutes a generalization because Larissa did see a relevant connection between the two.

Teacher: Oh my gosh. Why?

Larissa: Because it's, they're multiples of 4/5.

Julie: Because we found that the original was 12 and 15, okay, so we had to simplify that to 4/5. And so we know, and we tried out all of the ones that said yes—were multiples of 4/5.

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[3] Students pronounced ratios such as 5/6 as "five sixths."

Julie's remark shows that once the students had a list of correct same-speed pairs, each of which they had designated with a "yes," they tested whether all of the "yes" pairs were multiples of 4/5. This represents the generalizing action of *searching* for the same relationship, and by pointing out that all of the correct pairs were multiples of 4/5, Julie's reflection generalization was the *identification* of the common property across the correct pairs. They found this to be true in each case in their table, and the students justified their idea as follows:

Julie: Because the 12 and 15 simplified is 5 and 4.

Larissa: It works because, because it, because this one's 4 times bigger than this one and you need to do the opposite on the other ones to get 1.

Teacher: Why?

Larissa: Because you want to make them as equal as possible.

Teacher: What's 4 times bigger than what?

Larissa: Um, remember when we did that thing with the gears and we had to do the opposite on this one and this one [pointing to the cm and s in the table]? Like we had to do this one on this one and this one on this one? It's the same thing here.

Teacher: Why does that work?

Larissa: Because . . . you want to equal them out . . . so that . . . they can . . . this one [pointing to 4 sec] is 4/5 of this one [pointing to 5 cm]. So if you do 4/5 times 5, you get 4, which is this one [4 sec]. But that doesn't really help.

The students' appeal to symbolic rules for calculating equivalent fractions, in combination with a noticeable absence of any reference to the quantities in the situation, suggests that they were thinking of the symbols in absence of their connection to centimeters and seconds. Larissa also appeared to refer to a rule about making the numbers "equal," or balancing them out. The students' struggle in providing an explanation, combined with their focus on calculational rules rather than quantitative relationships, suggests that their proof scheme at this point was the external conviction symbolic scheme.

Dani then noticed that one of their correct pairs was not a multiple of 4/5: "Except for this one. 7.5 and 6. That won't be a multiple." When asked about this pair, Dani noted that it was related to the 15:12 pair in a different way:

Dani: You divide the 15 and 12 by 2.

Timothy: Yeah but . . .

Teacher: So does that fit with your multiples of 5 and 4 idea?

Dani: Yeah, sort of.

Timothy: Oh! You multiply whatever the centimeters are by 4/5 to equal the seconds. So it doesn't matter if they're multiples as long as, if you put the centimeters over the, or, the seconds over the centimeters it would equal the same as 4/5.

Teacher: So by multiples we don't necessarily mean whole-number multiples?

Timothy: It doesn't matter because it could also be fractions.

Timothy produced a reflection generalization, the *identification* of a general rule, after hearing Dani's idea. The students were further pressed to explain why this worked:

Teacher: Now I have not heard a justification why this works.

Julie: Because the numbers 12 and 15.
Timothy: 12 over 15 equals 4/5.
Dani: You can simplify and that's why. You can simplify 15 over 12.
Julie: You can simplify it down.
Teacher: So those are all forms of 12 and 15?
Julie: Yeah. It has to be a form of 12 and 15.
Teacher: Why?
Timothy: I don't know. It just works!

The students' struggle to explain Timothy's general rule remained at the level it was before. They still referred to symbolic rules for simplifying in absence of the associated quantity of speed; thus, they operated from the external conviction symbolic proof scheme.

Mechanisms for Change for Episode 1

Mechanism 1: Action/Reflection. Students' generalizing began at a fairly sophisticated level. They engaged in Type III generalizing, *extending*, when they operated on the 15:12 pair in order to generate new pairs. This action led to the *definition* of a class of same-speed pairs. However, their defined class was incomplete, containing only whole-number multiples. What occurred next was a second generalization pair, *relating* by connecting back to the gears situation and the *influence* of a prior idea, which led to a larger class of multiples of 5 and 4. Although this new defined class still does not include all same-speed pairs, it encompasses more pairs than the students were able to generate originally. Thus, the second action/reflection pair helped the students generate a more sophisticated notion of all same-speed pairs.

To check whether their idea about multiples of 5 and 4 was correct, the girls engaged in the generalizing action of *searching* for the same relationship across all of the pairs and were able to *identify* the property common to the correct pairs. This action/reflection pair aided in the clarification of the refined class of same-speed values. The action of *searching* helped the students solidify their idea about the refined class, which demonstrates how a different type of generalizing, in this case *searching*, can lead to a more sophisticated product, the *definition* of a larger and more accurate class of objects.

Finally, Timothy responded to the class discussion with a reflection generalization, the *identification* of a general rule. There is little evidence about what prompted this general rule for Timothy, but he appeared to react to 7.5 and 6, the one pair that was not a whole-number multiple of 4/5. Timothy's general rule could be very powerful, potentially allowing students to generate a more inclusive class of same-speed values. However, the students struggled to explain why it worked, operating under the external conviction symbolic proof scheme.

Episode 1 shows that a path of generalizing in distinct action/reflection pairs allows students to make new inferences. Although the students began by determining that multiples of 15 and 12 would result in same-speed values, they ended by realizing that there are more same-speed values than they had originally identified. The students created a new set of mathematical objects based on the 5:4 pair. By relating and implementing a prior idea, the students created the larger class, which culminated in Timothy's generation of a potentially powerful rule.

Mechanism 2: Focus. Students focused on numbers and patterns, which they represented in tabular form. This focus allowed them to *extend* their reasoning by operating on a same-speed pair and to *define* a class of same-speed values. However, given that the students remained focused on numbers in a way that appeared divorced from the quantities of centimeters and seconds, or the phenomenon of speed, it is not surprising that they operated within the symbolic proof scheme. The use of this proof scheme reflected the students' focus on symbols. Episodes 2 and 3 will demonstrate a change in the students' focus, which in turn will influence the type of generalizations and proof schemes they employ.

Episode 2: Timothy Makes Connections to the Quantities

The next day's problem (see Figure 7) followed up on the previous discussion by specifically requiring students to justify their reasoning. Timothy drew a graph (see Figure 8) and asked, "Does this count as a drawing, showing that . . . they're going up the same rate?" When asked what he thought, Timothy replied, "It does because it's saying like, here's 15 and 12, and see, Frog can go on any one of these." The teacher suggested further questions for Timothy to consider:

Teacher: Why do the points fall in a line?

Timothy: Because no matter what, they're all going to follow the same rule.

Teacher: What does each point represent?

Timothy: Represents the centimeters per second. Or . . . well, centimeters per second. 2.5 cm per 2 s.

Teacher: So this point represents 2.5 cm in 2 s. What does this point [indicating a different point] represent?

Timothy: Uh, 20 cm per 16 s.

A. Last time you figured out that multiples of 5 cm in 4 s work. What kind of multiples? Whole-number multiples? Do fractions or decimals work? Explain why multiples of 5 cm in 4 s work to produce a same speed.

B. Draw a picture of Clown walking 15 cm in 12 s and Frog walking 5 cm in 4 s. Show how these two scenarios represent the same speed.

C. Draw a picture of Clown walking 5 cm in 4 s and Frog walking 7.5 cm in 6 s. Show how these two scenarios represent the same speed.

Figure 7. Follow-up justification problem.

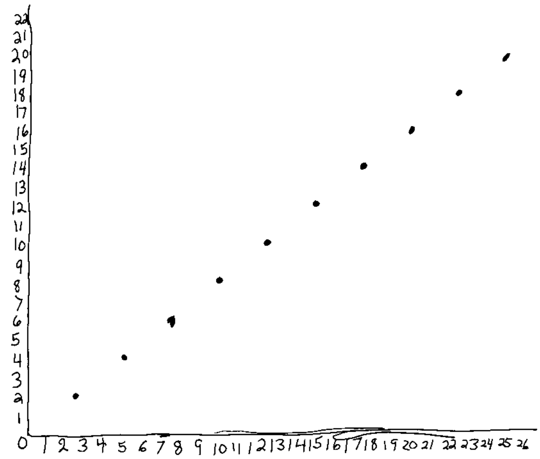

Figure 8. Timothy's graph showing same-speed points.

Teacher: So what's the same about these two points?

Timothy: They're all the same centimeters per second. The amount of centimeters that Frog's gonna run in a certain amount of time.

The line of questioning was designed to explicitly direct Timothy's attention to sameness across the points. This likely afforded Timothy's generalizing action of *searching* for the same relationship, because he had to think about what was the same about the points (2.5, 2) and (20, 16). Timothy then stated that all of the points represent the centimeters Frog travels in a particular time (centimeters per seconds). Thus, Timothy generated a reflection generalization in his *identification* of the property common to all of the points.

When asked about the slope of the line, Timothy identified it as 4/5:

Timothy: So that means, since the slope is 4/5, this [gesturing down the *y* column] is 4/5 of this [gesturing across the *x* column]. Basically, whatever *y* is, is 4/5 of whatever *x* is.

Teacher: Oh. And how does that 4/5 slope relate to what you're figuring out with the speed?

Timothy: Because let's see . . . for every centimeter it goes, it's going like 4, er, yeah 4/5 of a second I think. And it's the other way around? I don't know. Every centimeter goes . . . yeah. Every centimeter it's going it's 4/5 of a second. So . . . for 15 cm, 4/5 of 15 would equal 12. So since there's 15 cm, 4/5 of 15 cm, for every centimeter it's going 4/5 of a second, or 12 s.

Timothy's first statement that "whatever *y* is, is 4/5 of whatever *x* is" represents a reflection generalization—an *identification* of a general pattern. When asked to connect this pattern to the speed, he produced a different reflection generalization. Timothy's statement about what happens every centimeter is an *identification* of a continuing phenomenon, because he identifies the dynamic relationship between centimeters and seconds. In addition, he has made a connection between 4/5 and the notion of walking and speed. The 4/5 pattern now has meaning in terms of the quantities in the situation.

When the group convened, Timothy shared his ideas with the other students:

Timothy: Since the line was linear, and since we had already figured out that whatever the amount of centimeters that people walked, that was how many, that was 4/5, 4/5 of that was how many seconds it took. And so the graph's showing the different amount of centimeters and different amount of time they could have taken. And since, as long as you did like one of these [gesturing to the line] or beyond that, you would always end up having them go along at the same speed. Because since 15 and 12 is also on that line. And so . . . since 15 and 12 is also on that line, and you do anything else that's on that line, you'll be going at the same speed. Just one of them will stop at a certain time.

There is some evidence to suggest that Timothy's reasoning demonstrated the use of the transformational proof scheme. Given his statement that "you would always end up having them go along at the same speed" and his explanation that one can stop elsewhere on the line and still represent the same speed, it appears that Timothy was transforming images in an anticipatory manner; he could imagine any given point as representing the given speed. Furthermore, Timothy's statement that "you do anything else that's on the line" suggests that he anticipated that any point on the line, not just the points that he had drawn from his table, would represent the same speed.

To explain the meaning of slope to the other students, Timothy communicated the reflection generalization: "The slope means that whatever x goes up by . . . 4/5 of that is how much y goes up by." This may appear similar to Timothy's prior statements, but there is an important difference. He no longer limits his statement to the fact that y is 4/5 of x or that the seconds are 4/5 of the centimeters. Timothy now understands that any increase in y will be 4/5 of the same increase in x. This is a general pattern, as before, but it is a different, more powerful general pattern; it applies to all linear functions of the form $y = \mathrm{m}x + \mathrm{b}$ rather than just functions of the form $y = \mathrm{m}x$.

Mechanisms for Change for Episode 2

Mechanism 1: Action/Reflection. On the previous day, Timothy had *identified* a general rule: You multiply the centimeters by 4/5 to get the seconds. By creating a graph and focusing on what was the same about the points, Timothy's generalizing action of *searching* for the same relationship led to another reflection generalization—the *identification* of a general pattern: "Whatever y is, is 4/5 of whatever x is." Once he tried to connect this pattern to speed, Timothy realized that for every centimeter the Frog walked, it took 4/5 of a second. This reflection generalization is different from the others. First, it is an *identification* of a continuing phenomenon, so it is stated in a different form. But more important, it was the first time one of the students stated a connection between the general pattern and the speed situation. The 4/5 now carried a quantitative meaning for Timothy. Certainly it might have before, but this was the first time he stated this connection explicitly. Additionally, his struggle in producing the statement suggests that Timothy was making the connection as he spoke it.

Another cycle of generalizing and justifying produced Timothy's final reflection generalization, the statement of a different general pattern conveying the meaning of slope. This is arguably more sophisticated than the generalizations that preceded it, because now Timothy understood that not only is seconds proportional to centimeters, but the *change* in seconds in proportional to the *change* in centimeters. This understanding could be more helpful when students later address $y = \mathrm{m}x + \mathrm{b}$ situations, and Timothy may now be better prepared to extend his understanding to linear situations that are not directly proportional.

Mechanism 2: Focus. Episode 2 shows how a focus on quantities can result in generalizations that connect number patterns to situations and to justifications with the transformational proof scheme. For Timothy, the teacher's prompt to connect his general pattern to the speed situation allowed him to create new generalizations, such as an *identification* of a continuing phenomenon and a different general pattern tied to the transformational proof scheme. Before this shift in focus, all of the students' reflection generalizations were statements of general principles and definitions of classes, and the proof schemes associated with their attempts to justify remained at the external symbolic levels.

Timothy's attempt to explain *why* multiples of 4/5 resulted in same-speed values, rather than justify the fact alone, could have been the catalyst to develop a connection between his pattern and the quantitative relationship. He responded to three important questions: (1) Why do the points fall in a line? (2) What does each point represent? and (3) What is the slope of this line? These questions prompted Timothy to think about why the slope was 4/5, and why 4/5 was also the pattern and rule that he had previously identified. He was able to connect the pattern to the relationship between the y-values and x-values of the points in a way that was general: "Whatever y is, is 4/5 of whatever x is." By trying to explain why, Timothy was encouraged to focus on the relationship between the points in a way that did not depend on a particular point or pair of values.

Mechanism 3: Generalizations that promote deductive reasoning. Timothy's generalizations about the pattern between y and x and its relationship to speed appear to have been connected to the transformational proof scheme. He was able to logically deduce, as a result of operating on the constant ratio, that the Clown would walk 1 cm in 4/5 s. In order to do so, Timothy could have anticipated what the Frog's unit rate would have to be, given that the relationship between time and distance was 4/5. Furthermore, there is evidence that Timothy could appreciate the generality of the 4/5 relationship, as he was able to state it in several different ways and with different numeric examples. Although the conclusion that Timothy acted with the transformational proof scheme is tentative, it is likely that the ways in which he generalized afforded such reasoning. Specifically, Timothy's action of *searching* for the same relationship across the points on his graph could have strengthened his understanding of the invariant multiplicative relationship between centimeters and seconds. By focusing on this sameness, Timothy may have lifted his reasoning to a level at which he could think about the relationship between centimeters and seconds represented by each point.

When Timothy focused on a particular point, such as (2.5, 2), he could explain its meaning by stating, "This means he went 2.5 cm in 2 s." However, when he had to attend to what was the same across two points, such as (2.5, 2) and (20, 16), the former statement no longer held for both points. What is the same about the first and the second points? "They're all the same centimeters per second." Thus, Timothy had to think about the relationship between centimeters and seconds in a general way, considering how each point represented a different instantiation of the same relationship.

This generalizing action appeared to push Timothy to think about the ratio relationship in a way that was not dependent on one particular pair, thus encouraging the consideration of generality. The actions prompted by searching for the same relationship also supported other aspects of the transformational proof scheme, such as transforming images in an anticipatory manner in order to predict that any point on the line would represent the same speed. Similarly, Timothy's *identification* of a continuing phenomenon could have further strengthened his understanding of the idea that the ratio between centimeters and seconds remains constant as the Frog continues to move.

Mechanism 4: Influence of deductive reasoning on generalizing. The episode shows how Timothy's deductive argument could have enabled the *identification* of a new general principle, a general pattern. Although the students had previously identified general patterns, Timothy's new general pattern was the first generalization that referred to a ratio of increases rather than a direct ratio. Timothy's attention to the ratio of increases occurred through the act of explaining how his graph represented the same speed values. By developing this explanation, Timothy transformed images in an anticipatory manner, which could have encouraged a shift in focus from the ratio between x and y to the ratio of increases between x and y. The very act of justifying within the transformational proof scheme afforded the new generalization.

Furthermore, through his explanation to the other students about *why* the graph demonstrated that the characters walked the same speed, Timothy was able to develop the new inference. He realized that the ratio of the change in seconds to the change in centimeters was 4/5, which was represented as the slope of the graph. In each case, it was through the struggle to explain why multiples of 4/5 resulted in same-speed values that Timothy was able to develop a deductive argument and subsequently make new generalizations.

Episode 3: Larissa and Maria Connect to Quantities

Unlike Timothy, Larissa and Maria did not produce a graph. In the drawing they shared with the rest of the class (see Figure 9), the top number line represents Frog's journey and the bottom represents Clown's journey. The boxed numbers represent the number of seconds corresponding to the number of centimeters on each number line; thus the boxed "4.0" directly above and below the "5" on each number line shows that at 5 cm, both the Frog and the Clown had traveled for 4 seconds. In the following dialogue, which occurred prior to Timothy's explanation, Larissa explains how their picture showed that Frog and Clown walked the same speed.

Maria: Okay, we figured out that every 8, .8 s, no every second you go .8 cm.

Timothy: I think it's the other way around.

Dora: Every centimeter you go .8 s.

Timothy: Because that would explain 15 cm, 12 s. Because the smaller amount of seconds.

Maria: Okay. And, in 4 s the Frog reached 5 cm, and that was the speed of the Clown. In 12 s, 15 cm. In 8 s, the 10 cm. In 4 s, he reached the 5 cm.

Teacher: Excellent. Now Larissa, can you explain how this picture shows that Frog and Clown are going the same speed?

Larissa: Because for the, when they're at 4, both of them are at 4 s. But since the frog stops, he's finished. So he's finished at 4 s. But the clown keeps going and from 0 to 5 it jumped 4 s, from 5 to 10, and from 5 to 10 it also jumped 5 cm and 4 s. And from 10 to 15, it jumped 5 cm and also 4 seconds. So the proportion stays the same throughout the whole thing even though Frog stopped.

Through the process of developing a picture to justify the same-speed idea, Maria produced the reflection generalization of an *identification* of a continuing phenomenon: For every 1 cm the clown walked, it took 0.8 s. Although Maria stated it incorrectly, her written work prior to this discussion showed the correct relationship, which suggests that she misspoke. Thus, Maria produced a new statement about the speed situation, a statement the students had not previously constructed.

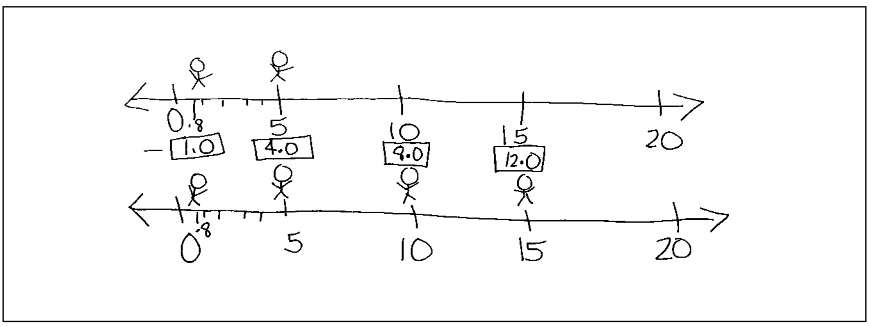

Figure 9. Larissa and Maria's diagram of Clown and Frog walking.

Larissa's explanation reveals elements of the transformational proof scheme, because she could imagine the Frog completing his journey at the same proportion as the Clown's completed journey. When subsequently asked if she could generalize her argument to any two characters walking the same speed, Larissa said, "The . . . proportion will stay the same. . . . If the two objects are walking the same speed, then the proportion throughout . . . their walking will stay the same even if one of the objects stops." Through discussing the picture and her justification, Larissa could now state a more general idea. She engaged in the generalizing action of *extending* by removing the particulars, because she had extended the idea of keeping the same proportion to any same speed pair. Larissa had also produced the reflection generalization that if the speed is the same, the proportion will remain the same: a global rule.

Mechanisms for Change for Episode 3 and General Discussion

Mechanism 1: Action/Reflection. Maria and Larissa identified generalizations that they had not previously considered. Maria produced a statement of a continuing phenomenon, and this generalization, like Timothy's, made sense of the speed situation where the girls' prior generalizations did not. It is difficult to ascertain what actions could have led to this reflection generalization because the only evidence available is their drawings. Maria's drawing (see Figure 9) was the culmination of three attempts. One possibility is that while struggling to create drawings, the students engaged in the generalizing action of *extending* by continuing, because they repeated the 5 cm:4 s unit three times on their paper in order to produce the number lines. In addition, the girls appear to have engaged in the generalizing action of *extending* by operating, because their final drawing showed that they partitioned the number line, which represents centimeters, into 5 equal parts. In order to partition their drawing, they would have had to mentally partition the seconds into 4 equal parts, then divide 5 cm by 4 in order to obtain 0.8 cm for 1 s. Maria marked the .8 cm on the number line, correctly showing at what point on the number line the Clown had traveled for 1 s. This generalizing action of *extending* could have contributed to the reflection generalization of the *identification* of a continuing phenomenon.

One more action/reflection cycle portrays further increased sophistication. When specifically asked to generalize her argument to any same-speed situation, Larissa engaged in the action of *extending* by removing particulars. Her prior generalizing actions, as well as her attempts to provide a reasonable explanation of the same-speed phenomenon, enabled her to engage in this final action. She was able to produce an *identification* of a global rule as a result: If two objects walk the same speed, the proportion (of distance to time) will remain the same. This is the type of global rule that is valued by educators, particularly because it is a more general statement. Larissa had now developed an inference about what same speed means, and her inference was not restricted to one particular problem. Thus, for each student, her action/reflection cycles allowed them to make new inferences about the speed situation, produce a general global rule, and produce statements connected to the transformational proof scheme, as discussed below.

Each of the three episodes shows different ways in which engaging in action/reflection cycles of generalization can contribute to the development of more sophisticated ideas. Although initial generalizations may have been limited or incorrect, subsequent cycles built on previous attempts to develop more broad and powerful generalizations. The episodes further demonstrate that the chain of students' generalizing does not occur haphazardly but instead in ways that allow students to bootstrap their reasoning into more sophisticated structures.

Mechanism 2: Focus. As in Episode 2, Episode 3 demonstrates that the students shifted their focus from number patterns to quantitative relationships. Before this shift in focus, the girls' reflection generalizations were *identifications* of general principles and *definitions* of classes, and their proof schemes associated with their attempts to justify remained at the external symbolic levels. After the shift, we see that the students produced *identifications* of continuing phenomena and justifications with the transformational proof scheme. Attending to the relationship between centimeters and seconds also helped the students make new inferences about the problem.

One of the catalysts for the shift in focus was again the need to explain *why* 5 cm in 4 s was the same speed as 15 cm in 12 s. Similar to Episode 2, the struggle to provide an explanation pushed the students to appeal to quantities, because appealing to patterns in the numbers produced justifications that the students sensed were ineffective. As the students struggled to develop an explanation, their focus changed, they produced different generalizations about new inferences, and they ultimately developed more sophisticated arguments.

Episodes 2 and 3 both showed that the students' shift in focus from number patterns to quantitative relationships afforded different types of generalizing and justifying. Once students focused their attention on the quantities, they began to generalize about relationships between quantities. One might argue that this type of evolution in students' reasoning could occur even without a shift in focus. However, this phenomenon did not occur at any time through the course of the teaching experiment, suggesting that the shift in focus played an important role in students' evolution in reasoning.

Mechanism 3: Generalizations that promote deductive reasoning. The possible generalizing actions of *extending* by operating and continuing, and the reflection generalization of an *identification* of a continuing phenomenon, supported Larissa's explanation via the transformational proof scheme. By *extending*, Larissa and Maria expanded the 5 cm:4 s ratio into a more general structure. Furthermore, they engaged in anticipatory, goal-oriented acts. Maria and Larissa knew that they wanted to determine a unit rate, and they were able to divide the ratio appropriately to obtain that rate. The resulting statement of a continuing phenomenon could also have supported Larissa's transformational proof scheme, because Larissa had shifted from thinking about multiples of 5:4 as producing same-speed values to thinking about iterating a 5 cm:4 s journey multiple times. Thus, she focused on the dynamic relationship between quantities, and was able to transform the image of Frog walking 5 cm in 4 s to Clown continuing his journey to a total of 15 cm in 12 s.

Episodes 2 and 3 demonstrated three types of generalizing that promoted the transformational proof scheme: the generalizing action of *searching for relationships*, the generalizing action of *extending*, and the reflection generalization of an *identification* of a continuing phenomenon. These were the three types tied to the use of the transformational proof scheme through analysis of the entire teaching experiment. The students' actions suggested that searching for the same relationship was not only connected with the use of the transformational proof scheme but may also have promoted its use. By turning their attention to relationships, students constructed new mathematical objects, such as ratios. Through repeated reasoning with the construction of new mental objects as relationships between existing objects, students honed their ability to operate on objects. In addition, as students searched for the same relationship, they anticipated that a certain relationship would remain stable throughout the data. Students' searching actions were also goal oriented, because they entered the search with the aim of developing a stable relationship. Therefore, the act of searching for a relationship incorporates many of the attributes that constitute the transformational proof scheme—goal-oriented, anticipatory operations on objects.

Similarly, the generalizing action of *extending* could promote the use of the transformational proof scheme. Because *extending* requires a student to expand his or her reasoning to incorporate nonpresent mathematical objects, he or she must both construct the relationship or pattern to be extended and anticipate how it could be extended. This could encourage students to operate on mathematical objects in a goal-oriented manner, further promoting an evolution in both their generalizing and justifying.

Finally, producing a statement of a continuing phenomenon appeared to be connected to the use of the transformational proof scheme precisely because it involves a focus on a dynamic relationship between quantities and the identification of a property that extends through time. In order to identify this property, students often had to transform images in an anticipatory, goal-oriented nature. Statements of continuing phenomena did not spontaneously appear for students. Often it was the process of justifying that supported their development; this phenomenon is discussed in more detail in Mechanism 4.

Mechanism 4: Influence of deductive reasoning on generalizing. Larissa produced a justification with the transformational proof scheme. Then, when pushed to further generalize, Larissa *extended* her reasoning by removing particulars and ultimately identified a global rule about same-speed values. Her act of justifying, particularly in a way that employed both goal-oriented, anticipatory actions and transformations of images, helped Larissa generalize further. Because one of the requirements of the transformational proof scheme is that students appreciate the generality aspect of a conjecture, engaging in this type of justification appears to afford further generalizing.

As discussed previously, two major types of reflection generalizations appeared after students justified with the transformational proof scheme: *identifications of general principles*, such as algebraic or global rules, and *identifications of continuing phenomena*. These episodes illustrate that the relationship between generalizing and justifying is not uni-directional. Students do not produce a generalization, justify it, and then move on. Instead, the act of justifying itself can push students' reasoning forward in ways that encourage further generalizing. When this act is connected to the transformational proof scheme, students' reasoning can progress in a manner that evolves toward increasingly sophisticated generalizations.

Final Remarks on the Episodes

The three episodes, taken together, demonstrate how the students' generalizing and justifying activities acted in concert to afford an evolution in their reasoning. The students began by relating the speed situation to the gears situation and by generalizing about number patterns. They developed a generalization that multiples of 5 cm:4 s would result in the same speed, but the students remained at the symbolic and empirical inductive proof scheme levels as they struggled to produce justifications. The teacher's prompt for the students to explain why encouraged a shift in focus from number patterns to quantities. This shift in focus prompted students to begin generalizing differently; they made statements of continuing phenomena and connected their generalizations to the quantitative relationships in the situation. The students were able to justify their new generalizations with the transformational proof scheme, which in turn encouraged the development of more powerful general principles related to linearity. The episodes were chosen as a coherent narrative showing all four mechanisms interacting to support increasingly sophisticated generalizations and justifications. Although some mechanisms may operate more effectively as stand-alone supports than others, it is the interaction between the four mechanisms that constitute a support for increased sophistication, rather than each mechanism occurring in isolation from the others.

DISCUSSION AND CONCLUSION

The four mechanisms for change are not hierarchical, but instead interact in complex ways to support more powerful acts of generalizing and justifying over time. These mechanisms emerged in part as a way to explain the phenomenon that students' generalizing and justifying were rarely distinct and separable acts. Instead, each activity influenced the other in a cyclical manner as students' reasoning evolved over time. The nature of this interaction between generalizing and justifying highlights the developmental importance of students' initial, limited general statements and proofs. Although correct algebraic generalizations and deductive forms of proof remain a critical instructional goal, this study suggests that students' incorrect, nondeductive generalizations and proofs may serve as an important bridge toward this goal.

Given the growing emphasis on proof in the middle grades, understanding which types of generalizing activities can support powerful justifications will be critical in helping educators design more effective lessons. For the teaching experiment participants, three types of generalizations were closely tied to reasoning with the transformational proof scheme: (1) searching for relationships, (2) extending one's reasoning, and (3) producing statements of continuing phenomena. The inclusion of problem situations that necessitate these actions may provide a fruitful setting to encourage the development of deductive reasoning skills as students are encouraged to explain why their generalizations make sense.

Results from the teaching experiment also address the role of justification as a support for generalizing. Recent pedagogical recommendations encourage teachers to present tasks in which algebra students find and generalize patterns (NCTM, 2000). However, as Lannin (2005) reminded us, "Developing algebraic understanding through patterning activities creates considerable difficulties as students move from a focus on particular examples toward creating generalizations" (p. 232). Furthermore, although these recommendations carry the assumption that generalizing patterns will constitute sufficient support for producing appropriate justifications, research demonstrates the widespread phenomenon of students using empirical justification to prove their generalizations (Hoyles, 1997; Knuth & Elliott, 1998; Lannin, 2005).

The fourth mechanism for change suggests that a more productive approach to proof instruction may challenge the typical generalization/proof sequence. Students in the study initially engaged in generalizing activities that were at times limited, partially incorrect, or otherwise unproductive. However, as they attempted to explain their generalizations and create increasingly deductive justifications, students were able to revisit their generalizing actions, build on them, and ultimately construct ones that were more powerful. The students' engagement in increasingly sophisticated generalization/justification cycles suggests that teachers might consider incorporating justification early into the instructional sequence, rather than expecting students to produce their final generalizations before moving on to proof. The role of proof could therefore be viewed as a way to help students generalize more effectively, rather than as an act that necessarily follows generalization.

In order to facilitate early engagement with justification and proof, teachers should consider emphasizing problems that allow for appropriate justification. In the context of linear function, this would mean de-emphasizing situations in which data are contrived or inexact, in favor of situations presenting linear data that students can investigate, manipulate, and make sense of. Although approximate data can constitute a powerful problem situation, particularly in terms of highlighting the role of mathematical models for making sense of messy real-world phenomena, these problems may be better reserved until after students have had opportunities to engage in the justifying acts that support the development of powerful generalizations about linearity.

Another feature of the results emphasized the role that students' mathematical focus plays in influencing the nature of their generalizing, which in turn affected the justifications they developed. This result suggests that problem situations that encourage a focus on relationships between quantities instead of number patterns or procedures alone could support the type of generalizing activity that encourages the development of powerful justifications, and vice versa. Curricular materials emphasizing quantitatively rich situations may provide a more fruitful setting to encourage productive generalizing and justifying. Teachers also play an important role in helping their students focus attention on quantities and the language of quantitative relationships. Although students often attend to number patterns alone, even within the context of a quantitatively rich problem, teachers can intervene to draw students' attention back toward the quantitative referents. In addition, teachers can incorporate the language of quantities into the classroom discussion by asking students to shift from pattern descriptions to phenomenon descriptions. Because students' interactions with problem situations matter as much as the situations themselves, the teacher's role is critical in choosing appropriate problems, shaping classroom discourse, posing questions requiring a focus shift, introducing ideas that emphasize relationships, and otherwise encouraging the type of focus shown to promote more effective generalizing and justifying.

The students in the study worked in a technology-based, small-group environment in which they were able to explore two quantitative situations in depth. Their experiences differed from the typical classroom, which merits future work examining the applicability of these findings to a wider range of instructional settings. Although a small-scale teaching experiment cannot, by its nature, produce widely generalizable conclusions, it can offer an initial framework for examining the ways in which generalizing and justifying mutually influence one another. The four mechanisms of change provide a way for researchers to make sense of the evolution of students' reasoning. They enable accounts of how generalizing and justifying are related activities, how they mutually influence one another, and how they can work together to support more sophisticated reasoning over time. Methodologically, the use of Harel and Sowder's (1998; Harel, 2006) proof scheme taxonomy, in combination with Ellis' (in press) generalization taxonomy, provides a way to examine students' mathematical generalizations and justifications in other content domains. Both taxonomies can inform the researcher about the actions students engage in as they generalize and prove, even if those actions are not ones typically valued by educators or researchers. The mechanisms of change show that students can, and do, move from less productive actions to more powerful generalizing and justifying behaviors over time. Thus, the actions that may be considered unacceptable from an expert's perspective could be the very ones that ultimately support more appropriate outcomes. By better understanding what students see as general and convincing, researchers and teachers can better help students move toward more powerful acts of generalizing and justifying.

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